THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Homework 2 solution
1st December 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. In order to compute the area, we just have to find out the angle A at the vertex $\frac{i}{\sqrt{3}}$ since the other two angles at the ideal points are 0. Let's consider the hyperbolic straight line connecting -1 and $\frac{i}{\sqrt{3}}$, it is a circular arc whose center is lying on the straight line $x = -1$ since it is perpendicular to the unit circle at -1. Let $c = -1 + yi$ be its center, then it is equidistant to 0 and $\frac{i}{\sqrt{3}}$. So $y^2 = 1 + (y - \frac{1}{\sqrt{3}})^2$. This gives $0 = 1 - \frac{2y}{\sqrt{3}} + \frac{1}{3}$. Therefore $y = \frac{2\sqrt{3}}{3}$. Now the angle of the tangent line of the circular arc at $\frac{1}{\sqrt{3}}$ is just given by the angle made by the radial lines from c to -1 and from c to $\frac{i}{\sqrt{3}}$. This is given by
$$\mathrm{Arg}\frac{\frac{i}{\sqrt{3}} - c}{-1 - c} = \mathrm{Arg}\frac{\frac{\sqrt{3}}{3}i + 1 - \frac{2\sqrt{3}}{3}i}{-\frac{2\sqrt{3}}{3}i} = \mathrm{Arg}(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = \arctan(\sqrt{3}) = \frac{\pi}{3}$$
By symmetry, the tangent of the hyperbolic straight line connecting 1 and $\frac{i}{\sqrt{3}}$ would make the same angle with the horizontal line. Therefore we have a Euclidean triangle with angles $A, \frac{\pi}{3}, \frac{\pi}{3}$. So it follows that A is also $\frac{\pi}{3}$. And the area of the hyperbolic triangle is $\pi - A = \frac{2\pi}{3}$.

2. The angle at the ideal point is 0, meanwhile two of the hyperbolic straight lines are classical straight line that meet at 0 at an angle of $\pi/2$. So it suffices to find out the remaining angle A at the vertex $2 - \sqrt{3}$ to determine the area. Following the same method as $Q1$, the center c of the hyperbolic straight line connecting i and $2 - \sqrt{3}$ is lying on the horizontal line $y = 1$, so the center $c = x + i$. And $i, 2 - \sqrt{3}$ are equidistant to this point. So $x^2 = (x - 2 + \sqrt{3})^2 + 1$. Simplifying this gives $2(2 - \sqrt{3})x = (7 - 4\sqrt{3}) + 1$ and therefore $x = \frac{4 - 2\sqrt{3}}{2 - \sqrt{3}} = 2$. Now the angle of the tangent line to the hyperbolic straight line at $2 - \sqrt{3}$ is given by
$$\frac{\pi}{2} - \mathrm{Arg}\frac{2 - \sqrt{3} - c}{i - c} = \frac{\pi}{2} - \mathrm{Arg}\frac{-\sqrt{3} - i}{-2} = \frac{\pi}{2} - \mathrm{Arg}(\frac{\sqrt{3}}{2} + \frac{i}{2}) = \frac{\pi}{3}$$
Notice that the the angle to the tangent now is just the angle A. So the area of the triangle is just $\pi - \frac{\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$. You should be careful that the way we obtain the tangent angles are different from $Q1$. You are advised to draw a picture to see how to find this angle.

3. By lecture 11 practice problem Q5, we know that the sum of interior angle of a polygon is $\sum_{i=1}^{n} \theta_i = (n-2)\pi - A$, where n is the number of sides. So the sum of exterior angle is $\sum_{i=1}^{n}(\pi - \theta_i) = 2n\pi - \sum_{i=1}^{n} \theta_i = 2n\pi - (n-2)\pi + A = 2\pi + A$.

4. By cosine rule I, $\cosh a = \cosh^2 a - \sinh^2 a \cos A$ so we have

$$\begin{aligned}
\cos A &= \frac{\cosh^2 a - \cosh a}{\sinh^2 a} \\
&= \frac{\cosh^2 a - \cosh a}{\cosh^2 a - 1} \\
&= \frac{(1+\sqrt{2})^2 - (1+\sqrt{2})}{(1+\sqrt{2})^2 - 1} \\
&= \frac{3 + 2\sqrt{2} - 1 - \sqrt{2}}{3 + 2\sqrt{2} - 1} \\
&= \frac{2+\sqrt{2}}{2+2\sqrt{2}} \\
&= \frac{1}{2} \cdot \frac{2+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Therefore $A = \frac{\pi}{4}$. And by symmetry all three angles are $\frac{\pi}{4}$, so the area is given by $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

5. (a) We can find out the two equal angles of the isosceles triangle from the area by $\frac{1}{2}(\pi - \frac{\pi}{6} - \frac{\pi}{2}) = \frac{\pi}{6}$. We can find out the side length of BC by the cosine rule II,

$$\begin{aligned}
\cosh BC &= \frac{\cos\frac{\pi}{6}\cos\frac{\pi}{2} + \cos\frac{\pi}{6}}{\sin\frac{\pi}{6}\sin\frac{\pi}{2}} \\
&= \frac{\sqrt{3}/2}{1/2} \\
&= \sqrt{3}
\end{aligned}$$

Now the inverse of $\cosh$ has an explicit formula from solving $y = \cosh x = \frac{e^x + e^{-x}}{2}$ in terms of x by solving the quadratic equation $e^{2x} - 2ye^x + 1 = 0$. Quadratic formula gives $e^x = y + \sqrt{y^2 - 1}$. So $x = \cosh^{-1}(y) = \ln(y + \sqrt{y^2-1})$.

In our case $BC = \ln(\sqrt{3} + \sqrt{(\sqrt{3})^2 - 1}) = \ln(\sqrt{3} + \sqrt{2})$.

(b) The shortest distance from C to AB is given by a hyperbolic straight line through C and perpendicular to AB, i.e. the altitude. Now this hyperbolic straight line L divides C into two smaller triangle. Applying sine rule on the altitude gives

$$\frac{\sinh L}{\sin\frac{\pi}{6}} = \frac{\sinh BC}{\sin\frac{\pi}{2}} = \sinh BC$$

So $\sinh L = \frac{1}{2}\sinh BC = \frac{1}{4}[e^{\ln(\sqrt{2}+\sqrt{3})} - e^{-\ln(\sqrt{2}+\sqrt{3})}] = \frac{\sqrt{2}}{2}$. Similar to inverse of $\cosh$, we have inverse $\sinh$ formula given by $\sinh^{-1}(y) = \ln(y + \sqrt{y^2+1}$. So $L = \ln(\frac{\sqrt{2}}{2} + \sqrt{\frac{1}{2}+1}) = \ln(\frac{\sqrt{2}}{2} + \sqrt{\frac{3}{2}})$. This gives a precise answer instead of a bound. One can see that it is bounded by $\ln(1+\sqrt{2})$ since $\frac{2}{2} < 1$ and $\sqrt{\frac{3}{2}} < \sqrt{2}$.